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A model for describing near-resonance oscillations in an elastic layer☆

A.G. Kulikovskii*, Ye.I. Sveshnikova

Moscow, Russia

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ABSTRACT

One-dimensional transverse oscillations in a layer of a non-linear elastic medium are considered, when one of the boundaries is subjected to external actions, causing periodic changes in both tangential components of the velocity. In a mode close to resonance, the non-linear properties of the medium may lead to a slow change in the form of the oscillations as the number of the reflections from the layer boundaries increases. Differential equations describing this process were previously derived. The equations obtained are hyperbolic and the change in the solution may both keep the functions continuous and lead to the formation of jumps. In this paper a model of the evolution of the wave patterns is constructed as integral equations having the form of conservation laws, which determine the change in the functions describing the oscillations of the layer as "slow" time increases. The system of hyperbolic differential equations previously obtained follows from these conservation laws for continuous motions, in which one of the variables is slow time, for which one period of the actual time serves as an infinitesimal quantity, while the second variable is the real time. For the discontinuous solutions of the same integral equations, conditions on the discontinuity are obtained. An analogy is established between the solutions of the equations obtained and non-linear waves propagating in an unbounded uniform elastic medium with a certain chosen elastic potential. This analogy enable discontinuities which may be physically realised to be distinguished. The problem of steady oscillations of an elastic layer is discussed.

When investigating transverse elastic waves, both continuous and discontinuous,[1–3] qualitatively new properties of their behaviour, due to the non-linearity of the equations and the anisotropy of the properties of the medium in planes of constant phase, were found. The most interesting results were obtained for quasi-transverse waves, in which, as is well known, the non-linearity of the relation between the stresses and strains has a cubic form, i.e., they are far weaker than in longitudinal or gas-dynamic waves. The wave must travel a much greater distance in order for its non-linear changes to be observed experimentally. To overcome this difficulty it is proposed to consider the problem of waves which propagate transverse to the elastic layer and which are alternately reflected from its boundaries. In this case the wave covers as large a distance as required while remaining within a region of finite size.

In order to discuss the change in the longitudinal component of the strain in the wave, the medium is assumed to be incompressible. It is assumed that the waves are produced by periodic motion of one of the boundaries in its plane. The other boundary of the layer is assumed to be fixed. For weak external actions, the non-linearity in the wave behaviour that arises is caused by the fact that the external action is close to resonance. The non-linear changes may accumulate with time and change the form of the oscillations. To describe the evolution of the oscillations a slow time was introduced earlier[4] and a hyperbolic system of equations was obtained. The real time acts as the second variable in these equations, i.e., in the role of the "spatial" variable. The hyperbolic system allows of the formation of discontinuities. Below, using the classical integral conservation laws for the model of an elastic medium, a model of the oscillatory process in the form of integral equations is constructed, from which, in the case of continuous motion, the above-mentioned differential equations are obtained by standard operations, while for the discontinuous motions relations are obtained on the front of the jump.

1. Specification of the elastic medium

Suppose plane uniform transverse waves propagate in both directions in a layer of constant thickness L of an incompressible non-linear elastic medium in a direction orthogonal to the boundaries. These waves are due to periodic oscillations of one of the boundaries

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* Corresponding author.
E-mail addresses: kulik@mi.ras.ru (A.G. Kulikovskii), sveshn@mech.mat.msu.su (Ye.I. Sveshnikova).

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in its plane, while the other boundary of the layer is fixed. The system will be described in Lagrange variables in a Cartesian system of coordiates $\{x_i\}$of the initial state. The direction of propagation of the oscillations is taken as the $x = x_3$ axis, the other two axes, x_1 and x_2, are parallel to the boundaries of the layer and, consequently, to the propagating wavefronts. The deformation is characterized by the components of the displacement gradient tensor $u_{ij} = \partial w_i/\partial x_j$, where w_i are the components of the displacement vector. As a consequence of the incompressibility $\partial w_3/\partial x = u_3 = \text{const}$, and we can take $u_3 = 0$. We will assume that the medium is homogeneous, i.e., its density ρ_0 and the temperature θ_0 are constant in the initial state. We will further put $\rho_0 = 1$. In one-dimensional motions of the incompressible medium, where $w_i = w_i(x,t)$, when a plane wave propagates, only the components of the shear deformation change $\partial w_i/\partial x = u_i(x,t)$, $i = 1,2$.

Hence, the elastic potential, i.e., the internal energy per unit volume of the medium, is given by the function $\Phi = \Phi(u_1, u_2, S)$, where S is the entropy per unit mass of the medium. It is assumed that the medium is slightly non-linear (the deformation is small) and possesses a slight anisotropy in the plane of the wavefront. Then, the function $\Phi(u_1, u_2, S)$ can be specified by an expansion in series in small deformations, retaining the first principal terms, representing the non-linearity and anisotropy[1,2]

$$\Phi = \frac{f}{2}(u_1^2 + u_2^2) + \frac{g}{2}(u_2^2 - u_1^2) - \frac{\kappa}{4}(u_1^2 + u_2^2)^2 + \theta_0(S - S_0) \tag{1.1}$$

The first term corresponds to a linear isotropic medium; its coefficient f differs only slightly from the shear modulus and determines the velocity of linear isotropic waves $c_0 = \sqrt{f}$. The isotropic non-linearity of the function Φ is represented by the term with the coefficient κ, which can have any sign. The coefficient g on the term having the anisotropic structure is taken to be small, so that the anisotropy and the non-linearity have the same order of magnitude. If we assume that the components of the deformation u_i and their change have an order of smallness ε while the elastic constants of the medium f and κ are finite, we must assume that the coefficient g has a value of the order of ε^2. The last term with the change in entropy $S - S_0$ is only important when investigating solutions with discontinuities, since in the assumed absence of heat exchange in the elastic medium in continuous processes the entropy does not change. For quasitransverse shockwaves it is well known,[1,2] that in jumps of small intensity the change in entropy on the front is three orders of magnitude less than the order of change of the remaining quantities. Hence, in expansion (1.1) the entropy is represented by the linear term with a constant coefficient.

The differential equations of one-dimensional motions of an incompressible elastic medium have the form[1,2]

$$\frac{\partial v_i}{\partial t} = \frac{\partial}{\partial x}\left(\frac{\partial \Phi}{\partial u_i}\right), \quad \frac{\partial u_i}{\partial t} = \frac{\partial v_i}{\partial x}, \quad i = 1, 2 \tag{1.2}$$

Here $v_i = \partial w_i/\partial t$ are the components of the velocity vector.

For waves of low intensity, propagating in a uniform background in one direction of the x axis, this system of four equations can be approximated, while retaining the required accuracy, by replacing it by a system of two equations[2,5]

$$\frac{\partial u_i}{\partial t} \pm \frac{\partial}{\partial x}\left(\frac{\partial \Phi_*}{\partial u_i}\right) = 0, \quad i = 1, 2 \tag{1.3}$$

in which the expression for Φ_* has the same structure as in (1.1), but with changed coefficients f_*, g_*, κ_*. The upper sign in these equations corresponds to waves propagating in the positive direction of the x axis, while the lower sign corresponds to waves propagating in the negative direction of the x axis.

Suppose the layer considered occupies the region between the planes $x = 0$ and $x = L$. The condition $v_i = 0$ is satisfied on the fixed boundary $x = 0$. The other boundary $x = L$ performs small oscillations in its plane, as given by the specified law $v_i = \psi_i(t)$. The elastic layer, when there is no non-linearity ($\kappa = 0$) and no anisotropy ($g = 0$), possesses natural oscillations, the period of which $T_0 = 2L/c_0$ is determined by the width of the layer L and the velocity of linear transverse waves $c_0 = \sqrt{f}$. The presence of small anisotropy and non-linearity makes the velocities of the two transverse waves c_1 and c_2 different, but the difference between the velocities c_0, c_1 and c_2 is of the order of ε^2.

The functions $\psi_i(t)$, specifying the amplitudes of the oscillation of the boundary, are assumed to be periodic with period T, close to T_0, differing from it by an amount of the order of ε^2. When investigating the continuous evolution of the waves[4] it was useful to introduce the quantity $a = 2L/T$, which has the dimension of velocity, and its difference from the characteristic velocities c_0, c_1 and c_2 is also of the order of ε^2. The quantities ψ_i are also assumed to be small compared with the characteristic values u_i. As was shown in Ref. 4, to achieve steady oscillations of the layer, the boundaries ψ_i supporting this mode of oscillation must be of the order of ε^3.

For the functions representing the oscillations on the mobile (right) boundary of the layer (for $x = L$), the following hyperbolic system of differential equations was obtained in Ref. 4,

$$\frac{\partial u_i}{\partial \tau} - \frac{\partial}{\partial \xi}\left(\frac{\partial \mathcal{F}}{\partial u_i}\right) = \frac{\psi_i(\xi)}{L} \tag{1.4}$$

in which one of the variables is the slow time τ while the second variable $\xi = at$ varies in the range $0 \le \xi \le 2L$ during one period of real time. The function $\mathcal{F}$, obtained previously in Ref. 4, depends both on the current values of u_i and on the values of u_i and $u_i u_j$ averaged over a period. An expression for this will be derived below (equality (4.1)).

If we put ψ_i equal to zero in Eqs (1.4), these equations acquire the form of Eqs (1.3), which describe waves propagating in the negative direction of the x axis, but with Φ_* replaced by $\mathcal{F}$.

For fixed values of the average quantities mentioned, the operator on the left-hand side of Eq. (1.4) is hyperbolic. As a consequence of this, Eqs (1.4) may have solutions which lead to the formation of discontinuities. The relations on the fronts of the discontinuity must be obtained from the integral equations corresponding to differential equations (1.4). These integral equations are obtained below.

2. Integral conservation laws and their modification

We will show how, starting from the integral equations of the mechanics of an elastic medium, one can obtain integral equations which express the conservation laws connecting the quantities when $x=L$. Equations (1.4) follow from these integral equations in the case of continuous solutions, while in the case of discontinuous solutions one obtains relations on the discontinuities.

The integral equations of the theory of elasticity for one-dimensional motions,[1,2] which hold for the arbitrary piecewise-continuous functions v_i and u_i, have the form

$$\frac{d}{dt}\int_{x_1}^{x_2} v_i dx + \left.\frac{\partial\Phi}{\partial u_i}\right|_{x=x_1} - \left.\frac{\partial\Phi}{\partial u_i}\right|_{x=x_2} = 0, \quad \frac{d}{dt}\int_{x_1}^{x_2} u_i dx + v_i|_{x=x_1} - v_i|_{x=x_2} = 0 \tag{2.1}$$

Here x_1 and x_2 are arbitrary constant quantities. The first group of equations expresses the law of conservation of momentum, while the second group expresses the relation between the deformations and the velocities of the medium. The law of conservation of energy in the assumed approximation is "detached" from the system and serves for the subsequent calculation of the change in entropy. It is therefore not used here. Equations (1.2) follow from Eqs (2.1) for the differentiable functions.

The integral equations corresponding to the approximate form of Eqs (1.3) for waves travelling in one direction and the conditions on the discontinuity which follow from them have the form

$$\frac{d}{dt}\int_{x_1}^{x_2} u_i dx + \left(\frac{\partial\Phi_*}{\partial u_i}\right)_{x=x_2} - \left(\frac{\partial\Phi_*}{\partial u_i}\right)_{x=x_1} = 0, \quad W[u_i] = \left[\frac{\partial\Phi_*}{\partial u_i}\right] \tag{2.2}$$

To obtain the integral equations for the new model, the accurate integral equations of the theory of elasticity (2.1) serve as the initial ones. Integrating them over an arbitrary time interval (t_1, t_2), we obtain

$$\left.\int_{x_1}^{x_2} v_i dx\right|_{t=t_2} - \left.\int_{x_1}^{x_2} v_i dx\right|_{t=t_1} - \left.\int_{t_1}^{t_2}\frac{\partial\Phi}{\partial u_i}dt\right|_{x=x_2} + \left.\int_{t_1}^{t_2}\frac{\partial\Phi}{\partial u_i}dt\right|_{x=x_1} = 0$$

$$\left.\int_{x_1}^{x_2} u_i dx\right|_{t=t_2} - \left.\int_{x_1}^{x_2} u_i dx\right|_{t=t_1} - \left.\int_{t_1}^{t_2} v_i dt\right|_{x=x_2} + \left.\int_{t_1}^{t_2} v_i dt\right|_{x=x_1} = 0$$

These equalities can be regarded as the equality to zero of the circulation over the contour of an arbitrary rectangle $x_1 \le x \le x_2$, $t_1 \le t \le t_2$ in the x, t plane of each of the vector fields

$$\mathbf{p}_i = \{p_{it}, p_{ix}\}, \quad \mathbf{q}_i = \{q_{it}, q_{ix}\}, \quad i = 1, 2$$

the components of which along the x and t axes have the form

$$p_{ix} = v_i, \quad p_{it} = \frac{\partial\Phi}{\partial u_i}, \quad q_{ix} = u_i, \quad q_{it} = v_i$$

Since the chosen rectangle is arbitrary, the circulation of each of the vectors $\mathbf{p}_i$ and $\mathbf{q}_i$ is equal to zero over any closed contour C in the x, t plane:

$$\oint_C (p_{ix}dx + p_{it}dt) = 0, \quad \oint_C (q_{ix}dx + q_{it}dt) = 0 \tag{2.3}$$

This representation of the equations enables us to calculate the change per period of the integrals of the quantities u_i and v_i. The calculations were carried out approximately, satisfying the assumed degree of accuracy.

We transform the integrand in expressions (2.3). Instead of the elastic potential Φ (1.1) we introduce the new function

$$F(u_1, u_2) = \Phi - a^2\frac{u_1^2 + u_2^2}{2}; \quad a = \frac{2L}{T} = \text{const} \tag{2.4}$$

The function F is the result of subtracting the linear isotropic part from the elastic potential, and hence it is of the order of ε^4, and its derivatives $F_i(u_k) = \partial F/\partial u_i$ are of the order of ε^3.

Instead of the functions u_i and v_i we will introduce the new variables

$$w_i^+ = v_i - au_i, \quad w_i^- = v_i + au_i \tag{2.5}$$

They are Riemann invariants of the left-hand sides of the system of equations

$$\frac{\partial v_i}{\partial t} - a\frac{\partial v_i}{\partial x} + a\left(\frac{\partial u_i}{\partial t} - a\frac{\partial u_i}{\partial x}\right) = \frac{\partial F_i}{\partial x}$$
$$\frac{\partial v_i}{\partial t} + a\frac{\partial v_i}{\partial x} - a\left(\frac{\partial u_i}{\partial t} + a\frac{\partial u_i}{\partial x}\right) = \frac{\partial F_i}{\partial x}; \quad i, k = 1, 2 \tag{2.6}$$

of the equivalent initial system (1.2). Here

$$u_k = \frac{1}{2a}(w_k^- - w_k^+), \quad v_k = \frac{1}{2}(w_k^- + w_k^+) \tag{2.7}$$

In the new variables, Eqs (2.6) take the form

$$\frac{\partial w_i^{\pm}}{\partial t} + \frac{\partial}{\partial x}(\pm a w_i^{\pm} - F_i) = 0, \quad i = 1, 2 \tag{2.8}$$

The boundary conditions for this system can also be written in terms of the function $w_i^{\pm}$

$$w_i^+ = -w_i^- \text{ when } x = 0, \quad w_i^+ + w_i^- = 2\psi_i \text{ when } x = L$$

Equations (2.3) can be written in terms of the function $w_i^{\pm}$

$$\oint_{C^-}((-aw_i^- - F_i)dt - w_i^- dx) = 0, \quad \oint_{C^+}((aw_i^+ - F_i)dt - w_i^+ dx) = 0 \tag{2.9}$$

Since the functions $F_i(u_k)$ themselves are of the order of ε^3, we will take as their arguments the solution of the linear problem, i.e., Eqs (2.6) or, correspondingly, (2.8) for $w_i^{\pm}$ with zero right-hand sides, in which we put $F_i = 0$. The solution of the linear problem has the form of travelling waves with constant velocities $\pm$ a

$$w_i^{0+} = \varphi_i\left(t - \frac{x}{a}\right), \quad w_i^{0-} = \vartheta_i\left(t + \frac{x}{a}\right)$$

Taking into account the smallness of the functions $\psi_i(t)$ of the external action, the boundary conditions of the linear problem are

$$w_i^{0+} = -w_i^{0-} \text{ when } x = 0 \text{ and } = L$$

Hence it follows that $\varphi_i = -\vartheta_i$ for the same values of the arguments, while the functions φ_i and ϑ_i are periodic with period T.

3. The non-linear change in the integral quantities over a period

In order to obtain integral equations analogous to (2.2) for the model describing slow changes in the amplitude and form of the oscillations, we need to calculate the change over a period T of the quantities $\int_{t_1}^{t_2} u_i dt$ and $\int_{t_1}^{t_2} v_i dt$ for an arbitrary time interval $[t_1, t_2]$.

For each of the two groups of equations (2.9) we choose its contour: C^- and C^+ respectively. For the first group we take as C^- the contour ADHE (see Fig. 1), consisting of the two characteristics AD and EH of the family $x + at = \text{const}$, along which the values of w_i^{0-} are preserved, and of the end sections AE and DH, belonging to the layer boundaries, where $w_i^{0+} = -w_i^{0-}$. The vertices of this quadlateral have coordinates $A(L,t_1)$, $D(0,t_1 + T/2)$, $E(L,t_2)$ and $H(0,t_2 + T/2)$. The distance between the points A and E along the time axis can have any value, including as small as desired. For the second group of equations (2.9), in the same way as C^+, we choose the contour BDHF, formed by the two characteristics of the family $x - at = \text{const}$ on which $w_i^{0+} = \text{const}$, and the end sections, which belong to the boundaries of the layer. The vertices B and F have coordinates $B(L,t_1 + T)$, $F(L,t_2 + T)$.

From the two equations of the first group (2.9) we obtain

$$a\int_A^E w_i^- dt - a\int_D^H w_i^- dt = \int_A^E F_i dt - \int_D^H F_i dt - \int_A^D F_i dt + \int_E^H F_i dt \tag{3.1}$$

Similarly, the second group of equations (2.9) gives

$$a\int_B^F w_i^+ dt - a\int_D^H w_i^+ dt = \int_D^H F_i dt - \int_B^F F_i dt - \int_D^B F_i dt + \int_H^F F_i dt \tag{3.2}$$

We add the equations of the first and second groups with the same subscripts i term-by-term, taking into account the fact that on the common section DH of the left boundary according to the boundary conditions $w_i^+ = -w_i^-$. Then the integrals along this section cancel out. Moreover, as was stated above, the functions φ_k and ϑ_k are periodic with period T. Hence, the functions $F_i(\varphi_k, \vartheta_k)$ have the same values on

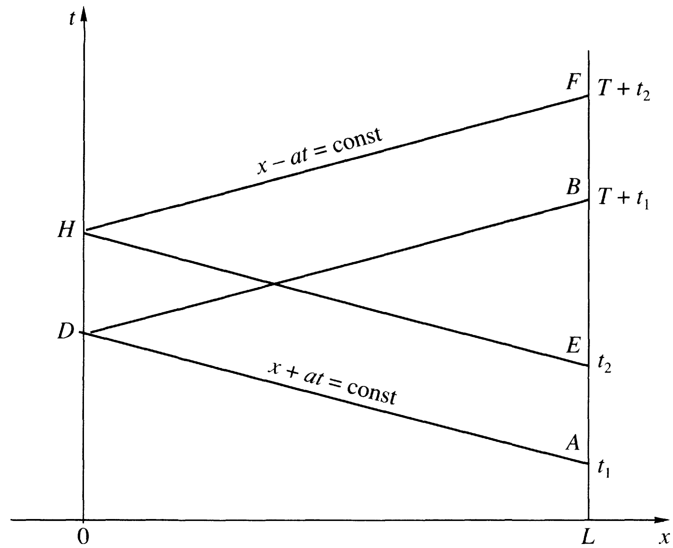

Fig. 1.

the sections AE and BF, shifted by a period T along the time axis. Hence, when Eqs (3.1) and (3.2) are added with an error of the order of ε^6, the integrals along sections AE and BF cancel out. As a result we obtain

$$\int_A^E w_i^- dt + \int_B^F w_i^+ dt = \mathcal{M}_i$$
$$a\mathcal{M}_i = -\int_A^D F_i dt + \int_E^H F_i dt + \int_H^F F_i dt - \int_D^B F_i dt \tag{3.3}$$

We will first convert the left-hand side of the first equation of (3.3). We add and substract $\int_A^E w_i^+ dt$. We obtain

$$\int_{t_1+T}^{t_2+T} w_i^+ dt - \int_{t_1}^{t_2} w_i^+ dt + \int_{t_1}^{t_2} (w_i^- + w_i^+) dt = \mathcal{M}_i$$

The first two terms give the change in the quantity $\int_{t_1}^{t_2} w_i^+ dt$ over a period T. If this difference is divided by T, the expression obtained can be treated as the derivative with respect to the slow time τ, for which one period T of real time serves as the unit of measurement. The slow time τ was also introduced earlier (Ref. 4). The integrands in the last term on the left-hand side of the last equality, by definition (2.7), represent the components of the velocity v_i on the right boundary of the layer, where, according to the boundary conditions $\upsilon_i = \psi_i(t)$. So the last equation takes the form

$$T\frac{d}{d\tau}\int_{t_1}^{t_2} w_i^+ dt + 2\int_{t_1}^{t_2} \psi_i(t) dt = \mathcal{M}_i \tag{3.4}$$

Similarly, adding and subtracting $\int_B^F w_i^- dt$ on the left-hand side of (3.3), we obtain

$$-T\frac{d}{d\tau}\int_{t_1}^{t_2} w_i^- dt + 2\int_{t_1}^{t_2} \psi_i(t) dt = \mathcal{M}_i \tag{3.5}$$

Adding and subtracting Eqs (3.4) and (3.5) between each other for each subscript i and taking equalities (2.7) into account, we return to the initial components of the velocity and deformation

$$\frac{d}{d\tau}\int_{t_1}^{t_2} u_i dt - \frac{2}{aT}\int_{t_1}^{t_2} \psi_i(t)dt = -\frac{\mathcal{M}_i}{aT} \tag{3.6}$$

$$\frac{d}{d\tau}\int_{t_1}^{t_2} \upsilon_i dt = 0 \tag{3.7}$$

We recall that $aT = 2L$, which simplifies the form of the coefficient of the second term. The integrands in Eqs (3.6) and (3.7) are evaluated on the right-hand boundary of the layer with $x = L$. On this boundary $\upsilon_i = \psi_i(t)$, and hence, in view of the assumed periodicity of the functions which define the shift of the boundary points, Eqs (3.7) are satisfied identically.

We will now obtain an explicit form of the expressions for $\mathcal{M}_i$, which were introduced by Eqs (3.3). To do this we integrate the functions $F_i(\varphi_k, \vartheta_k)$ over the sections indicated in the (x,t) plane (see the figure), maintaining the assumed accuracy. The functions $F_i = \partial F/\partial u_i$ are evaluated in terms of the specified elastic potential (1.1), (2.4):

$$F_i = \frac{\partial F}{\partial u_i} = g_i u_i - \kappa u_i(u_1^2 + u_2^2), \quad i = 1, 2$$

$$g_1 = f - g - a^2, \quad g_2 = f + g - a^2$$

The quantity a^2 was introduced so that $f - a^2 \sim \varepsilon^2$ and the coefficient of anisotropy of the medium g has the same order ε^2, and hence the new coefficients g_1 and g_2 are also of the order of ε^2, and both terms in the functions $F_i(u_k)$ are of the order of ε^3. Consequently, satisfying the limits of accuracy, when calculating F_i the quantities u_k can be replaced by their linear approximation in ε, i.e., we assume

$$u_k^0 = \frac{1}{2a}\left(\vartheta_k\left(t + \frac{x}{a}\right) - \varphi_k\left(t - \frac{x}{a}\right)\right)$$

Then

$$F_i = \frac{g_i}{2a}(\vartheta_i - \varphi_i) - \frac{\kappa}{8a^3}(\vartheta_i - \varphi_i)[(\varphi_1 - \vartheta_1)^2 + (\varphi_2 - \vartheta_2)^2]$$

When evaluating the integrals of these functions along the sections indicated in relations (3.3), we take into account the fact that along the sections AD and EH, along the characteristics of the family $x + at = \text{const}$, the functions ϑ_i are constant and equal to their values $\vartheta_i(A)$ and $\vartheta_i(E)$ at the corresponding points on the right boundary. Along the sections AD and EH the values of the functions φ_i are transferred along the characteristics of the other family ($x - at = \text{const}$) from the sections of the boundary $x = L$ $AB(t_1, t_1 + T)$ and $EF(t_2, t_2 + T)$ respectively. Hence, the integration along the sections AD and EH can be replaced by integration along the sections AB and EF of the t axis, where the integration is carried out in both cases over a time interval equal to the period T. Along the sections DB and HF, along the family of characteristics $x - at = \text{const}$, the functions φ_i are constant and equal to their values on the right boundary: on the section DB $\varphi_i = \varphi_i(B)$ and along the section HF $\varphi_i = \varphi_i(F)$. In view of the periodicity of the functions φ_i these values are equal to $\varphi_i(A)$ and $\varphi_i(E)$ respectively. Similarly, the quantities ϑ_i are transferred from the same sections AB and EF of the right boundary to the characteristics of the family $x + at = \text{const}$ and are integrated with respect to time over a period. After carrying out the integration, the functions ϑ_i are everywhere replaced by $-\varphi_i$, in accordance with the boundary conditions.

As a result we obtain expressions, in explicit form, for the right-hand sides of Eqs (3.6)

$$\frac{\mathcal{M}_i}{aT} = P_i(A) - P_i(E) \tag{3.8}$$

where

$$P_i = -\frac{1}{2a^2}\Big[g_i\varphi_i - \frac{\kappa}{4a^2}(\varphi_i^3 + \varphi_i\varphi_{3-i}^2 + 3\bar{\varphi}_i\varphi_i^2 + \bar{\varphi}_i\varphi_{3-i}^2 + $$
$$+ 2\bar{\varphi}_{3-i}\varphi_1\varphi_2 + 3\bar{\varphi}_{11}\varphi_i + \bar{\varphi}_{22}\varphi_i + 2\bar{\varphi}_{12}\varphi_{3-i})\Big], \quad i = 1, 2 \tag{3.9}$$

$$\bar{\varphi}_i = \frac{2}{T}\int_0^T \varphi_i dt, \quad \bar{\varphi}_{ij} = \frac{2}{T}\int_0^T \varphi_i\varphi_j dt; \quad i, j = 1, 2 \tag{3.10}$$

In order to substitute the expressions obtained for $\mathcal{M}_i$ into Eqs (3.6) we must revert from the functions φ_i to the initial components of the deformations u_i using formulae (2.7). Since the functions P_i themselves are of the order of ε^3 (as follows from their form), in expressions (2.7) it is sufficient to use the approximation for $w_i^{\pm}$, linear in ε, and this means that on the right boundary (where all the equations are written) we can assume that $\upsilon_{i0} = 0$ and we can put $\varphi_i = -au_i$ in the expressions for P_i. In order that the left-hand sides of Eqs (3.6) should take the standard form of the conservation laws, we make the replacement of variables $at = \xi$ (as was done earlier in Ref. 4), where the variable ξ with dimension of length varies over a single reflection cycle $0 \le \xi \le 2L$.

As a result we obtain equations, the left-hand side of which can be written in the form of integral conservation laws for quantities on the right-hand boundary of the layer $x = L$, taking into account the external actions, represented by the functions ψ_i

$$\frac{d}{d\tau}\int_{\xi_1}^{\xi_2} u_i d\xi - P_i(u_k)\big|_{\xi=\xi_2} + P_i(u_k)\big|_{\xi=\xi_1} = \frac{1}{L}\int_{\xi_1}^{\xi_2}\psi_i(\xi)d\xi \tag{3.11}$$

The expressions for the functions P_i in terms of u_k have the form

$$P_i = \frac{1}{2a}\Big\{g_i u_i - \frac{\kappa}{4}[(u_i+\bar{u}_i)^3 + (u_i+\bar{u}_i)(u_{3-i}+\bar{u}_{3-i})^2 + \\ +(3h_{ii}+h_{3-i,3-i})u_1 + 2h_{12}u_{3-i}]\Big\}, \quad i = 1, 2 \tag{3.12}$$

$$\bar{u}_i = \frac{2}{T}\int_0^T u_i dt, \quad h_{ij} = \frac{2}{T}\int_0^T u_i u_j dt - \bar{u}_i\bar{u}_j; \quad i, j = 1, 2 \tag{3.13}$$

The latter quantities are functions of τ.

4. The elastic potential of the equivalent medium

The differential equations and conditions on the discontinuity. The expressions $P_i(u_1, u_2)$ enable us to consider them as derivatives of a certain function $\mathcal{F}(u_1, u_2)$, i.e., $P_i = \partial\mathcal{F}/\partial u_i$. Considering the role which these derivatives $P_i = \partial\mathcal{F}/\partial u_i$ play in Eqs (3.11) of the conservation laws we can treat the function $\mathcal{F}(u_1, u_2)$ introduced in this way as an elastic potential of a certain medium, the properties of which enable us to describe the evolution of the oscillations. An explicit form of the expression for this function is obtained as a result of corresponding integration of Eqs (3.12)

$$\mathcal{F}(u_1, u_2) = \frac{\bar{f}}{2}(u_1^2+u_2^2) + \frac{\bar{g}}{2}(u_2^2-u_1^2) - \frac{\bar{\kappa}}{4}[(u_1+\bar{u}_1)^2 + (u_2+\bar{u}_2)^2]^2 - 2\bar{\kappa}h_{12}u_1u_2 \\ \bar{f} = \frac{f-a^2}{2a} - 2\bar{\kappa}(h_{11}+h_{22}), \quad \bar{g} = \frac{g}{2a} - \frac{\bar{\kappa}}{2}(h_{22}-h_{11}), \quad \bar{\kappa} = \frac{\kappa}{8a} \tag{4.1}$$

The function $\mathcal{F}$ is identical with that obtained previously in Ref. 4 when considering continuous processes using differential equations and occurs in Eq. (1.4).

System (3.11) takes the form where the left-hand side of the equations are identical with those of Eqs (2.2) for waves in an unbounded space, which propagate in one direction:

$$\frac{d}{d\tau}\int_{\xi_1}^{\xi_2} u_i d\xi - \frac{\partial\mathcal{F}}{\partial u_i}\bigg|_{\xi=\xi_2} + \frac{\partial\mathcal{F}}{\partial u_i}\bigg|_{\xi=\xi_1} = \frac{1}{L}\int_{\xi_1}^{\xi_2}\psi_i d\xi, \quad i = 1, 2 \tag{4.2}$$

In the case when derivatives of the functions $u_i(\xi, \tau)$ exist, we obtain the following differential equations from Eqs (4.2)

$$\frac{\partial u_i}{\partial\tau} - \frac{\partial}{\partial\xi}\left(\frac{\partial\mathcal{F}}{\partial u_i}\right) = \frac{1}{L}\psi_i(\xi) \tag{4.3}$$

the structure of the left-hand sides of which are similar to the structure of the left-hand sides of Eqs (1.3) for waves travelling in the negative direction in an unbounded elastic space. In the model obtained, the quantity $\xi = at$ (proportional to the time) serves as the spatial variable, which can only increase.

When the solution is piecewise-continuous, and the functions ψ are bounded, from Eq. (3.11) we obtain relations on the discontinuity which are the same as in the last equation of (2.2)

$$W[u_i] = [\partial\mathcal{F}/\partial u_i], \quad W = d\xi/d\tau \tag{4.4}$$

Note that the left-hand sides of Eqs (4.2)-(4.4) are identical with the left-hand sides of Eqs (1.3) and (1.4) if we replace Φ_* in them by $\mathcal{F}$. Consequently, Eqs (4.2)-(4.4) are equivalent to the equations which describe the propagation of waves in an elastic medium, the properties of which are determined by how the function $\mathcal{F}$ depends on u_1 and u_2. The quantities $\bar{f}$, $\bar{g}$ and $\bar{\kappa}$ occur in the coefficients of this relation, the first two of which are, in general, functions of τ. The values of these quantities are calculated by averaging the solutions over the period T. Over a single period the changes in these quantities are of the order of ε^3, which enables them to be assumed to be constant over this time. It should be noted that even if the initial elastic medium is isotropic ($g = 0$), as a result of the non-linear evolution the new medium may acquire wave anisotropy ($\bar{g} \neq 0$, if $h_{22} \neq h_{11}$) and lead to the occurrence in the equation of a term of the same order, which is also non-linear. Moreover, Eqs (4.2) and (4.3) include external actions, specified by the functions $\psi_i(\xi)$.

The fact that Eqs (4.4) are identical with the conditions on the discontinuities, which propagate on a uniform background in a medium with elastic potential $\mathcal{F}(u_1, u_2)$, enables us to apply the results of previous investigations[2,3] to this case. If the oscillations are unsteady in the slow-time mode, i.e., they depend on τ, the function $\mathcal{F}$ also depends on τ and the medium used for the comparison will also vary with τ. We can introduce the new variables $u_i^{\sim} = u_i + \bar{u}_i$. In these variables Eqs (4.3) remains as before with the exception of the fact that a term $\partial\bar{u}_i/\partial\tau$ is added to its right-hand side. Since $\bar{u}_i = \bar{u}_i(\tau)$ and $\partial/\partial u_i = \partial/\partial u_i^{\sim}$, the relations on the discontinuity (4.4) in this conversion retain their form. With this replacement $\mathcal{F}$ as a function of $u_i^{\sim}$ does not contain cubic terms, while the term of the fourth power is isotropic in the x_1, x_2 plane, i.e., it has a form for which the conditions on the discontinuity have been investigated in detail.[2]

Just as in the case of the propagation of perturbations in a uniform state, in many cases (and in particular, with the assumption of "vanishing viscosity") to obtain a discontinuity it is necessary to satisfy the conditions of a priori evolutionability, which consists of the fact that the characteristics of one of the families arrives as the discontinuity from both sides, whereas the characteristics of the other family passes through it. The discontinuities can then be divided into two types: fast and slow shock waves[2] respectively, the characteristics of the family of which arrive at the discontinuity from both sides.

Equations (4.2) and (4.3) show that the derivatives of u_i with respect to τ are continuous if the derivatives u_i with respect to ξ are continuous, and at the instant of time τ considered they depend, via the averaged quantities (3.13), on the preceding time interval of duration T. This enables us to consider the solution in the variable ξ in the region $0 \le \xi \le 2L$. The ends of this section can be identified, i.e., we must require the equality of u_i when $\xi = 0$ and for $\xi = L$ for all τ, with the exception of the instants of arrival at those points of the discontinuity of the solution described by equalities (4.4). Here Eqs (4.3) will be considered on a closed curve of length L without any boundary conditions.

5. Steady oscillations of the layer

We will consider steady oscillations when u_i is independent of τ. The case of one-dimensional transverse oscillations, when the deformation of the medium is characterized by a single variable, was investigated in Ref. 6. Since $\partial u_i/\partial\tau = 0$, Eqs (4.3) can be integrated with respect to ξ:

$$\partial\mathcal{F}/\partial u_i = -A_i(\xi), \quad \xi = at \tag{5.1}$$

The functions $A_i(\xi)$ are the primitives of $\psi_i(\xi)/L$, representing the displacements of the boundary divided by L. For a periodic solution to exist it is necessary that the functions A_i should be periodic, i.e., that the integral of $\psi_i(\xi)$ over a period should be equal to zero. If we assume $\mathcal{F}$ to be a known function of u_1 and u_2, Eqs (5.1) enable the possible values of u_i to be obtained independently for each ξ. When ξ changes, these values may change continuously and may undergo discontinuities, on which, according to relations (3), the following equalities must be satisfied

$$[\partial\mathcal{F}/\partial u_i] = 0 \tag{5.2}$$

For qualitative investigations and to construct examples of the oscillations we can specify the function $\mathcal{F}(u_1, u_2)$ in a certain way, obtain $u_1(\xi)$, calculate $\bar{u}_i$ and h_{ij} using relations (3.13), and then, using equalities (4.1), obtain the coefficients which define the elastic potential Φ of the oscillating medium. If, when formulating the problem, the properties of the elastic medium must be specified in a quite definite way, it becomes possible to choose the parameters of the function $\mathcal{F}(u_1, u_2)$ to obtain the required function $\Phi(u_1, u_2)$. Without considering this more complex problem here, we will confine ourselves to considering the motion, assuming the function $\mathcal{F}(u_1, u_2)$ to be known.

Equations (5.1) for each value of ξ determine the state of the medium with elastic potential $\mathcal{F}(u_1, u_2)$, the stresses in which along the x_1 and x_2 axes are equal to the current values of the functions A_1 and A_2. The problem of determining the possible states u_i, which satisfy Eqs (5.5), reduces to finding the stationary points of the function

$$N(u_1, u_2) = \mathcal{F}(u_1, u_2) + A_1u_1 + A_2u_2 \tag{5.3}$$

Since A_1 and A_2 are functions of ξ, the position of the stationary points will also depend on ξ. The coordinates of one of the stationary points of this function determine the current values of $u_1(\xi)$ and $u_2(\xi)$, which may vary continuously as ξ changes, and may undergo discontinuities which satisfy conditions (5.2).

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